Краткие сообщения ОИЯИ №9-85 JINR Rapid Communications No.9-85
УДК 530.145.530.12.

ON THE GAUGE-INVARIANT VARIABLES FOR NON-ABELIAN THEORIES

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Gauge invariant field variables are proposed for the case of non-Abelian field theories. The relation of these variables with the gauge-invariant strength tensor is found. It is shown that the Lorentz gauge condition formulated in terms of Mandelstam's contour derivatives takes place for new field variables and it serves as a secondary constraint according to Dirac's definition.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

О калибровочно-инвариантных переменных в неабелевых теориях

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В неабелевой калибровочной теории поля вводятся калибровочно-инвариантные полевые переменные. Найдена их связь с калибровочно-инвариантным тензором напряженности. Показано, что для введенных полевых переменных выполняется в качестве вторичной связи условие Лоренца, записанное в терминах контурных производных Мандельстама.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

In our previous paper /1/ we have formulated a gauge-invariant approach to quantum electrodynamics. It was shown that gauge-invariant field variables introduced in /1/ coincide with the usual fields taken in some gauge. The so-called inversion formulae that connect in a simple way gauge-invariant vector fields with the strength tensor $F_{\mu\nu}$ were found. It was shown that for these fields the Lorentz gauge condition takes place as a secondary constraint in accordance with Dirac's definition. In the present paper we give a generalization of the results of /1/ to the non-Abelian case.

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We define the field potential

$$B_\mu(x\,|\,\xi) = A_\mu(x) - \partial_\mu \int_\xi^x d\eta^\nu(a) A_\nu(\eta(a)) - \\ - ig \int_0^1 da\, a\, [A_\mu(\eta(a)), A_\nu(\eta(a))], \tag{1}$$

where $\eta(a) = \xi + a(x - \xi)$, $0 \le a \le 1$ and $A_\mu(x)$ is the non-Abelian vector field.

It is easy to see that in the Abelian case $B_\mu(x\,|\,\xi)$ coincides with the field of Fock's class [2,3].

Integrating by parts and with the help of the definition of the strength tensor

$$F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) - ig[A_\mu(x), A_\nu(x)] \tag{2}$$

we find a relation of the field $B_\mu(x|\xi)$ with $F_{\mu\nu}$

$$B_\mu(x\,|\,\xi) = \int_0^1 da\, a\, (x - \xi)^\nu F_{\mu\nu}(\xi + a(x - \xi)). \tag{3}$$

Formula (3) coincides with an inversion formula obtained in Fock's gauge $(x - \xi)^\mu A_\mu^F(x) = 0$. It should be noted that the strength tensor $F_{\mu\nu}$ is taken in an arbitrary gauge and not necessarily in Fock's gauge.

In [4] (see also [5]) the operator

$$U(x\,|\,c) = P \exp[-ig \int_{-\infty}^x d\eta^\nu A_\nu(\eta)] \tag{4}$$

has been introduced, where P means an ordering along the contour C. Now we perform a gauge transformation

$$A_\mu(x) \to A_\mu^\omega(x) = \omega(x) A_\mu(x) \omega^{-1}(x) + \frac{i}{g} \partial_\mu \omega(x) \omega^{-1}(x)$$

with $\omega(x) = U^+(x|C)$. Under this transformation $F_{\mu\nu}$ transforms into the gauge-invariant tensor $\mathcal{F}_{\mu\nu}(x|C) = U^+(x|c) F_{\mu\nu}(x) U(x|C)$ considered in [5], and the field $B_\mu(x|\xi)$, defined by (3), transforms into the gauge-invariant vector field

$$\mathcal{B}_\mu(x|C) = \int_0^1 da\, a\, (x - \xi)^\nu \mathcal{F}_{\mu\nu}(\xi + a(x - \xi)|C). \tag{5}$$

Tensor $\mathcal{F}_{\mu\nu}(x\,|\,C)$ obeys equality

$$\tilde{\partial}_\rho \mathcal{F}_{\mu\nu}(x\,|\,C) + \tilde{\partial}_\mu \mathcal{F}_{\nu\rho}(x\,|\,C) + \tilde{\partial}_\nu \mathcal{F}_{\rho\mu}(x\,|\,C) = 0$$

written in terms of Mandelstam's contour derivatives, that are defined as follows [4,5]:

$$\tilde{\partial}_\mu U(x\,|C) = \lim_{\Delta x \to 0} \frac{U(x + \Delta x|C') - U(x\,|\,C)}{\Delta x}, \tag{6}$$

where contours C and C′ differ only by a value of Δx. With the help of this equality it is possible to show that

$$\mathcal{F}_{\mu\nu}(x|C) = \tilde{\partial}_\nu \mathcal{B}_\mu(x;\ \xi|C) - \tilde{\partial}_\mu \mathcal{B}_\nu(x;\ \xi|C). \tag{7}$$

Thus, the relation of the gauge-invariant strength tensor $\mathcal{F}_{\mu\nu}(x|C)$ with the gauge-invariant vector field $\mathcal{B}_\mu(x;\xi|C)$ is analogous in form to the well-known relation that takes place in the Abelian case up to a substitution of ordinary derivatives by Mandelstam's contour derivatives.

With the help of (5) and taking into account antisymmetry of the tensor $\mathcal{F}_{\mu\nu}(x|C)$ and equation of motion $\tilde{\partial}^\mu \mathcal{F}_{\mu\nu}(x|C) = 0$ we find

$$\tilde{\partial}^\mu \mathcal{B}_\mu(x;\ \xi|C) = 0. \tag{8}$$

Formula (8) is nothing more but a secondary constraint (in accordance with Dirac's terminology /6/) and it has the meaning of generalization of the Lorentz condition for the non-Abelian case.

Now let us consider a generalization of Dirac's class of gauge-invariant fields /7/ for the non-Abelian case. We introduce the field

$$B_\mu(x|f) = A_\mu(x) - \int dy\, f^\nu(x-y) D_\mu A_\nu(y), \tag{9}$$

where $D_\mu = \frac{\partial}{\partial y^\mu} - ig[A_\mu, \ldots]$, is a usual covariant derivative and the function $f^\nu(x-y)$ obeys the conditions

$$\partial^\mu f_\mu(z) = \delta(z); \quad f^*_\mu(z) = f_\mu(z). \tag{10}$$

In the Abelian case the field (9) transforms into a field introduced by Dirac /7/. From (9) with the help of (10) and (2) the next formula follows

$$B_\mu(x|f) = \int dy\, f^\nu(x-y) F_{\mu\nu}(y). \tag{11}$$

By analogy with the previous case let us perform with the help of gauge transformation with $\omega(x) = U^+(x|C)$ a transition to the gauge-invariant variables $\mathcal{B}_\mu(x;f|C)$ and $\mathcal{F}_{\mu\nu}(x|C)$ connected by formula

$$\mathcal{F}_\mu(x;f|C) = \int dy\, f^\nu(x-y) \mathcal{F}_{\mu\nu}(y|C). \tag{12}$$

For the field (12) it is possible by analogy with the previous case to prove that the next formula holds

$$\mathcal{F}_{\mu\nu}(x|C) = \tilde{\partial}_\nu \mathcal{B}_\mu(x;f|C) - \tilde{\partial}_\mu \mathcal{B}_\nu(x;f|C) \quad (13)$$

and the condition

$$\tilde{\partial}^\mu \mathcal{B}_\mu(x;f|C) = 0 \quad (14)$$

takes place. It appears as a secondary constraint and has the meaning of the generalization of the Lorentz gauge condition for the non-Abelian case.

In conclusion it should be mentioned that a local phase transformation for spinor fields, that is consistent with a gauge transformation of the vector fields with $\omega(x) = U^+(x|C)$, leads to the gauge-invariant variables

$$\Psi(x|C) = \Gamma(U^+(x|C))\,\Psi(x) \quad (15)$$

where the matrix Γ, as usually, connects the adjoint and fundamental representations of the Lie groups.

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Received on May 17, 1985.